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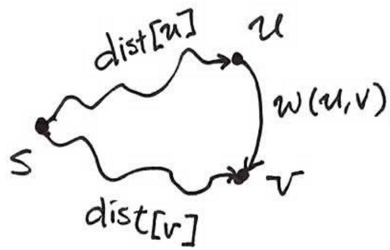
Bellman-Ford

- shortest paths w/ negative weight edges
- detects/finds negative cost cycles
- slow

Bellman-Ford

Idea: repeatedly relax edges until no new shortest paths found



$\text{dist}[u] + w(u,v) < \text{dist}[v]$?

Relax(u,v)
  if $\text{dist}[u] + w(u,v) < \text{dist}[v]$ then
    $\text{dist}[v] = \text{dist}[u] + w(u,v)$
    $\text{pred}[v] = u$

Each iteration of Bellman-Ford relaxes every edge in the graph exactly once.

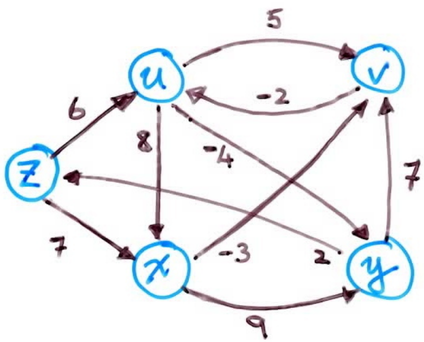
If there are no neg. cost cycles, how many iterations do we need?

Consider a shortest path p from s to v

$$v_0 = s \to v_1 \to v_2 \to v_3 \to \cdots \to v_{r-1} \to v_r = v$$

shortest path $s \rightsquigarrow v_i$ "found" before or during i^{th} iteration

Once a shortest path has been found, additional relaxing won't change dist[v_i]



(u,v)	5	(x,y)	9
(u,x)	8	(y,v)	7
(u,y)	-4	(y,z)	2
(v,u)	-2	(z,u)	6
(x,v)	-3	(z,x)	7

d[]	init	1	2	3	4	final
u	∞	6		2		2
v	∞		~~11~~ 4			4
x	∞	7				7
y	∞		2		-2	-2
z	0					0

π[]	init	1	2	3	4	final
u	nil	z		v		v
v	nil		~~x~~ x			x
x	nil	z				z
y	nil		u		u	u
z	nil					nil

Bellman-Ford (G, s)

① foreach $v \in V$, dist[v] = ∞, pred[v] = nil

② loop V-1 times

for each edge $(u,v) \in E$, Relax(u,v)

if dist[u] + w(u,v) < dist[v]
dist[v] = dist[u] + w(u,v)
pred[v] = u

③ /* Neg. cost cycle detection */

for each edge $(u,v) \in E$

shouldn't happen if no neg. cost cycles

if dist[u] + w(u,v) < dist[v] then
return (false)

means graph is bad

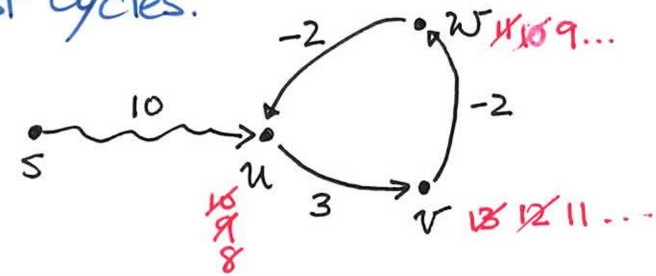
return true

Negative cost cycles:

If G does not have negative cost cycles,
all shortest paths are found within V-1 iterations
So, no additional improvement possible.

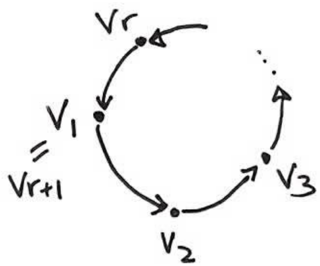
⇒ algorithm returns "true".

Need to show algorithm will return "false" when
G does have negative cost cycles.



Proof that Bellman-Ford detects negative cost cycles.

Let $v_1, v_2, \ldots, v_r, v_{r+1} = v_1$ be a neg. cost cycle.



$$\sum_{i=1}^{r} w(v_i, v_{i+1}) < 0$$

Proof by contradiction

Suppose that for all i, $1 \le i \le r$, we have

$$dist[v_i] + w(v_i, v_{i+1}) \ge dist[v_{i+1}]$$

no need to relax

Then, $\displaystyle\sum_{i=1}^{r} dist[v_i] + \sum_{i=1}^{r} w(v_i, v_{i+1}) \ge \sum_{i=1}^{r} dist[v_{i+1}]$

We have

$$\sum_{i=1}^{r} dist[v_i] + \sum_{i=1}^{r} w(v_i, v_{i+1}) \geq \sum_{i=1}^{r} dist[v_{i+1}]$$

$$\sum_{i=1}^{r} dist[v_i] = dist[v_1] + dist[v_2] + \ldots + dist[v_r]$$

$$\sum_{i=1}^{r} dist[v_{i+1}] = dist[v_2] + dist[v_3] + \ldots + dist[v_r] + dist[v_{r+1}]$$

$$= dist[v_2] + dist[v_3] + \ldots + dist[v_r] + dist[v_1]$$

$v_1 = v_{r+1}$

are equal

Hence $\sum_{i=1}^{r} w(v_i, v_{i+1}) \geq 0$.

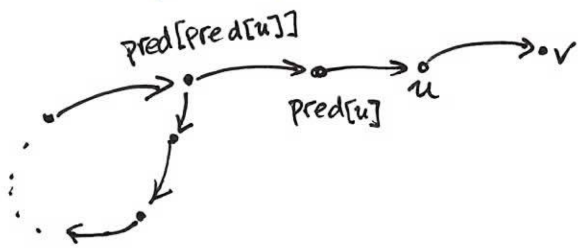
This contradicts $v_1, v_2, \ldots, v_r, v_{r+1}$ being a negative cost cycle. $\boxtimes$

How to find negative cost cycle?

① Let (u,v) be an edge found in the extra iteration where

$$\text{dist}[u] + w(u,v) < \text{dist}[v]$$

② Follow pred[u] back, until a vertex is repeated. (This forms a cycle.)



need to prove this has to happen

③ Cycle found will have negative cost.

need to prove this too.

Simple modification:

Bellman-Ford (G,s)

① for each $v \in V$, dist[v] = ∞, pred[v] = nil

instead of V-1

② loop V times

for each edge $(u,v) \in E$, Relax (u,v)

③ If any dist[] values change in the last iteration, return false

This change ensures that an edge that gets relaxed in the V^{th} iteration will also update the predecessor.
For example, if the graph has one negative cost cycle that includes all the vertices, we want pred[s] to be updated.

Claim 1: Following pred[] back from u will result in a cycle.

Pf: (by contradiction)

Suppose not. There are only V vertizes so after V steps back, some vertex must repeat, unless a nil pointer is encountered.

o.w. $dist[u] + w(u,v) = \infty$

Since $dist[u] \neq \infty$, the vertex with nil predecessor must be the source vertex s. Then we have the path

$$s = v_1 \to v_2 \to v_3 \to \cdots \to v_r = u$$

where $v_i = pred[v_{i+1}]$.

$$s = v_1 \to v_2 \to v_3 \to \cdots \to v_{r-1} \to v_r = u \to v$$

← not one of the v_i's

note: $r \le V-1$

Now, pred[s] = nil means dist[s] = 0 and s was never assigned a predecessor. (You can change predecessors, but not to nil.) So, dist[s] has been fixed since the beginning.

Then, dist[v_2] would have been discovered (and fixed) in iteration #1 when edge (s, v_2) was relaxed.

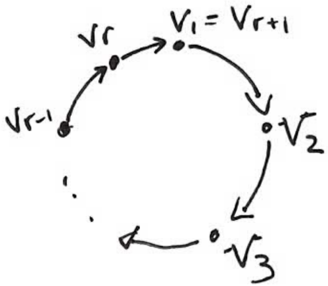
Similarly, dist[v_i] is fixed from iteration #($i-1$) on.

Then, dist[u] is fixed from iteration #($V-2$) on. ← because $u = v_r$ and $r \le V-1$

If dist[u] did not change in iteration #($V-1$), then Relax(u, v) cannot have any effect in the extra iteration used for neg. cycle detection. $\Rightarrow\Leftarrow$

Claim 2: Cycle found has negative cost.

Pf: Let $v_1, \ldots, v_r$ be the vertices in the cycle.



$v_i = \text{pred}[v_{i+1}]$

Let v_i be the vertex that was last to have dist[ ] change.

important!

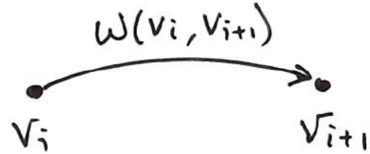
Defn: $\text{dist}^t[u]$ = dist[u] value at time t.

t_i = time when dist[v_i] was last changed

Then, $w(v_i, v_{i+1}) + \text{dist}^{t_{i+1}}[v_i] = \text{dist}^{t_{i+1}}[v_{i+1}]$

because v_{i+1} was relaxed at time t_{i+1}

WARNING: SUBTLE POINT



At time t_{i+1}, v_{i+1} was relaxed, so at time t_{i+1}

$$w(v_i, v_{i+1}) + dist^{t_{i+1}}[v_i] = dist^{t_{i+1}}[v_{i+1}].$$

that means $t_{i+1} < t_i$

Later, $dist[v_i]$ might decrease, if v_i is relaxed.

note the change from t_{i+1} to t_i

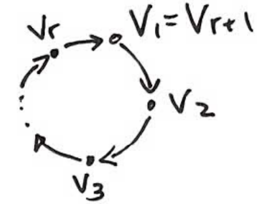
But, we can still claim:

$$w(v_i, v_{i+1}) + dist^{t_i}[v_i] \leq dist^{t_{i+1}}[v_{i+1}].$$

because dist[] values can only go down.

We know $w(v_i, v_{i+1}) + dist^{t_i}[v_i] \le dist^{t_{i+1}}[v_{i+1}]$ (*)

Now, $v_r = pred[v_1]$.



v_1 relaxed at time t_1

So,

$$dist^{t_1}[v_1] = w(v_r, v_1) + dist^{t_1}[v_r]$$

$$= w(v_r, v_{r+1}) + dist^{t_1}[v_r]$$

$$= w(v_r, v_{r+1}) + dist^{t_r}[v_r]$$

v_1 was last to change

$$\ge w(v_r, v_{r+1}) + w(v_{r-1}, v_r) + dist^{t_{r-1}}[v_{r-1}]$$ by (*)

$$\ge w(v_r, v_{r+1}) + w(v_{r-1}, v_r) + w(v_{r-2}, v_{r-1}) + dist^{t_{r-2}}[v_{r-2}]$$ by (*)

$$\vdots$$

$$\ge \left[\sum_{i=2}^{r} w(v_i, v_{i+1})\right] + dist^{t_2}[v_2]$$

special case for v_2

equals, not $\ge$

$$= \left[\sum_{i=2}^{r} w(v_i, v_{i+1})\right] + w(v_1, v_2) + dist^{t_2}[v_1]$$

(v_1, v_2) relaxed at time t_2, so $dist^{t_2}[v_2] = w(v_1, v_2) + dist^{t_2}[v_1]$.

$$\mathrm{dist}^{t_1}[v_1] \geq \left[\sum_{i=1}^{r} w(v_i, v_{i+1})\right] + \mathrm{dist}^{t_2}[v_1]$$

$$\Rightarrow \quad \sum_{i=1}^{r} w(v_i, v_{i+1}) \leq \mathrm{dist}^{t_1}[v_1] - \mathrm{dist}^{t_2}[v_1]$$

$$< 0$$

because we know dist$[v_1]$ decreased at time t_1 and we picked v_1 so t_1 is largest. Thus, $t_1 > t_2$ and

$$\mathrm{dist}^{t_1}[v_1] < \mathrm{dist}^{t_2}[v_1]$$

Thus, $v_1, \ldots, v_r, v_1$ is a negative cost cycle.

$\boxtimes$